4.7 Applications Involving Exponential Functions

Warm Up:

1. The volume of water in a container increases by 8% every day. If the initial volume is 100ml, what will the volume be in 3 days? $b = 1.08$

$$V = 100(1.08)^3$$
$$\doteq \underline{125.97 \text{ ml}}$$

Exponential Growth

2. The value of a vehicle depreciates by 20% every year. If the initial value of the vehicle is $20000, what will the value be in 5 years? $b = 0.8$

$$\text{Value} = 20000(0.8)^5$$
$$\doteq \underline{\$6553.60}$$

Exponential Decay

Given an equation of the form $y = ab^x$, Exponential Growth occurs when $b > 1$, Exponential Decay occurs when $0 < b < 1$. 'a' represents the initial value.

Ex

2. Complete the table.

	Function	Exponential Growth or Decay?	Initial Value	Growth or Decay Rate
a)	$V(t) = 20(1.02)^t$	growth	20	2%
b)	$P(n) = (0.8)^n$	decay	1	20%
c)	$A(x) = 0.5(3)^x$	growth	0.5	200%
d)	$Q(w) = 600\left(\frac{5}{8}\right)^w$	decay	600	$\frac{3}{8}$ or 37.5%

A 200 g sample of radioactive polonium-210 has a *half-life* of 138 days. This means that every 138 days, the amount of polonium left in a sample is half of the original amount. The mass of polonium, in grams, that remains after t days can be modelled by $M(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{138}}$.

a) Determine the mass that remains after 5 years.

b) How long does it take for this 200 g sample to decay to 110 g?

Key Ideas

- The exponential function $f(x) = ab^x$ and its graph can be used as a model to solve problems involving exponential growth and decay. Note that
  - $f(x)$ is the final amount or number
  - a is the initial amount or number
  - for exponential growth, $b = 1 +$ growth rate; for exponential decay, $b = 1 -$ decay rate
  - x is the number of growth or decay periods

Need to Know

- For situations that can be modeled by an exponential function:
  - If the *growth rate* (as a percent) is given, then the base of the power in the equation can be obtained by *adding* the rate, as a decimal, to 1. For example, a growth rate of 8% involves multiplying repeatedly by 1.08.
  - If the *decay rate* (as a percent) is given, then the base of the power in the equation is obtained by *subtracting* the rate, as a decimal, from 1. For example, a decay rate of 8% involves multiplying repeatedly by 0.92.
  - One way to tell the difference between growth and decay is to consider whether the quantity in question (e.g., light intensity, population, dollar value) has increased or decreased.
  - The units for the growth/decay rate and for the number of growth/decay periods must be the same. For example, if light intensity decreases "per metre," then the number of decay periods in the equation is measured in metres, too.

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~~#2~~

Knowledge

1. Evaluate the following: *(9 marks - 1 mark each)*

a) $\frac{2}{5^{-2}}$ $= 2 \cdot \frac{1}{5^{-2}}$ $= 2 \cdot \left(\frac{1}{5}\right)^{-2}$ $= 2\left(\frac{5}{1}\right)^{2} \nearrow 2(25) = \underline{\underline{50}}$	b) $\left(\frac{3}{6}\right)^{-1}$ $= \left(\frac{6}{3}\right)^{1}$ $= \frac{6}{3} \nearrow = \underline{\underline{2}}$	c) $(-8)^{\frac{5}{3}}$ $= \left(\sqrt[3]{-8}\right)^{5}$ $= (-2)^{5}$ $= \underline{\underline{-32}}$
d) $21(7^{-2})$ $= 21\left(\frac{1}{7^2}\right)$ $= \frac{21}{49}$ $= \underline{\underline{\frac{3}{7}}}$	e) $27^{\frac{2}{3}}$ $= \left(\sqrt[3]{27}\right)^{2}$ $= 3^{2}$ $= \underline{\underline{9}}$	f) $\left(\frac{5x^2+7z}{2y^2+9z^{\frac{1}{2}}}\right)^0$ $= \underline{\underline{1}}$
g) $(3^3+2^2)^{-1}$ $= \left(\frac{3^3+2^2}{1}\right)^{-1}$ $= \left(\frac{1}{3^3+2^2}\right)^{1}$ $= \frac{1}{27+4}$ $= \underline{\underline{\frac{1}{31}}}$	h) $(\sqrt[3]{3^5})(\sqrt[3]{3^4})$ $= \left(3^{\frac{5}{3}}\right)\left(3^{\frac{4}{3}}\right)$ $= 3^{\frac{5}{3}+\frac{4}{3}}$ $= 3^{\frac{9}{3}}$ $= 3^{3}$ $= \underline{\underline{27}}$	i) $\frac{3^8 \times 4^{-3}}{4^{-5} \times 3^7}$ $= \frac{(3^8)(4^{-3})}{(4^{-5})(3^7)}$ $= (3^{8-7})(4^{-3-(-5)})$ $= 3^1(4^2)$ $= 3(16)$ $= \underline{\underline{48}}$

3. Simplify the following expressions. Leave all answers with positive exponents: *(4 marks - 2 marks each)*

a) $(64x^6)^{-\frac{2}{3}} \div (16x^4)^{-\frac{1}{2}}$

$= (64^{-\frac{2}{3}})(x^6)^{-\frac{2}{3}} \div (16^{-\frac{1}{2}})(x^4)^{-\frac{1}{2}}$

$= \left(\frac{1}{64}\right)^{\frac{2}{3}} x^{-\frac{12}{3}} \div \left(\frac{1}{16}\right)^{\frac{1}{2}} x^{-\frac{4}{2}}$

$= \frac{1}{(\sqrt[3]{64})^2} x^{-4} \div \frac{1}{4} x^{-2}$

$= \frac{1}{4^2} \cdot \frac{1}{x^4} \div \frac{1}{4} \cdot \frac{1}{x^2}$

$= \frac{1}{16x^4} \div \frac{1}{4x^2}$

$= \frac{1}{16x^4} \times \frac{4x^2}{1}$

$= \frac{4\cancel{x^2}}{16\cancel{x^4}\,x^2}$

$= \frac{1}{4x^2}$

b) $(16x^8y^{-0.5})^{0.25}$

$\rightarrow (16x^8y^{-\frac{1}{2}})^{\frac{1}{4}}$

$= \sqrt[4]{16}\,x^2\,y^{-\frac{1}{8}}$

$\frac{2x^2}{y^{\frac{1}{8}}}$

a) $\left(\frac{16}{81}\right)^{-\frac{3}{4}}$

$= \left(\frac{81}{16}\right)^{\frac{3}{4}}$

$= \frac{(\sqrt[4]{81})^3}{(\sqrt[4]{16})^3}$

$= \frac{3^3}{2^3}$

$= \frac{27}{8}$

b) $\sqrt{25^3} + (25^{\frac{1}{3}} \times 25^{\frac{1}{6}})$

$= 25^{\frac{3}{2}} + 25^{\frac{1}{3}+\frac{1}{6}}$

$= (\sqrt{25})^3 + 25^{\frac{2}{6}+\frac{1}{6}}$

$= 5^3 + 25^{\frac{3}{6}}$

$= 125 + 25^{\frac{1}{2}}$

$= 125 + \sqrt{25}$

$= 125 + 5$

$= 130$

Solving Exponential Equations - Practice

1. Solve.

a) $2^x = 16$
b) $3^x = 27$
c) $2^x = 128$
d) $5^x = 125$
e) $4^y = 256$
f) $729 = 9^x$
g) $(-3)^x = -27$
h) $(-2)^x = -32$
i) $(-5)^r = 25$
j) $81 = (-3)^x$
k) $-2^x = -16$
l) $-4^y = -64$
m) $-5^x = -625$
n) $(-1)^x = 1$
o) $(-1)^m = -1$

2. Solve.

a) $7^{w-2} = 49$
b) $3^{x+4} = 27$
c) $2^{1-x} = 128$
d) $4^{3k} = 64$
e) $5^{3r-1} = 25$
f) $-81 = -3^{2x+8}$
g) $4^{x-1} = 1$
h) $3^{2-2x} = 1$
i) $(-1)^{2r} = 1$

3. Solve and check.

a) $6^{x+3} = 6^{2x}$
b) $2^{x+3} = 2^{2x-1}$
c) $3^{2y+3} = 3^{y+5}$
d) $2^{4x-7} = 2^{2x+1}$
e) $7^{5d-1} = 7^{2d+5}$
f) $3^{b-5} = 3^{2b-3}$

4. Solve.

a) $16^{2x} = 8^{3x}$
b) $4^t = 8^{t+1}$
c) $27^{x-1} = 9^{2x}$
d) $25^{2-r} = 125^{2r-4}$
e) $16^{2p+1} = 8^{3p+1}$
f) $(-8)^{1-2x} = (-32)^{1-x}$

5. Solve and check.

a) $2^{x+5} = 4^{x+2}$
b) $2^x = 4^{x-1}$
c) $9^{2q-6} = 3^{q+6}$
d) $4^x = 8^{x+1}$
e) $27^{y-1} = 9^{2y-4}$
f) $8^{x+3} = 16^{2x+1}$

6. Solve and check.

a) $5^{4-x} = \frac{1}{5}$
b) $10^{y-2} = \frac{1}{10\,000}$
c) $6^{3x-7} = \frac{1}{6}$
d) $3^{3x-1} = \frac{1}{81}$
e) $5^{2n+1} = \frac{1}{125}$
f) $\frac{1}{256} = 2^{2-5w}$

1 c)
$$2^x = 128$$
$$2^x = 2^7$$
$$\therefore \underline{x = 7}$$

2 e)
$$5^{3r-1} = 25$$
$$5^{3r-1} = 5^2$$
$$\therefore 3r - 1 = 2$$
$$3r = 3$$
$$\therefore \underline{r = 1}$$

2 g)
$$4^{x-1} = 1$$
$$4^{x-1} = 4^0$$
$$\therefore x - 1 = 0$$
$$\underline{x = 1}$$

4 f)
$$(-8)^{1-2x} = (-32)^{1-x}$$
$$((-2)^3)^{1-2x} = ((-2)^5)^{1-x}$$
$$(-2)^{3(1-2x)} = (-2)^{5(1-x)}$$
$$\therefore 3(1-2x) = 5(1-x)$$
$$3 - 6x = 5 - 5x$$
$$-6x + 5x = 5 - 3$$
$$-x = 2$$
$$\underline{\underline{x = -2}}$$

7. Solve and check.

a) $4^x = 8$
b) $64^x = 16$
c) $(-8)^y = -2$
d) $9^{-x} = 3$
e) $2^{9x} = \frac{1}{8}$
f) $9^{6x} = \frac{1}{27}$
g) $2^x = 16^4$
h) $2^{-2g} = 32$
i) $9^{2y+1} = 27$

8. Solve and check.

a) $9^{x+1} = 27^{2x}$
b) $16^y = 64^{2y-1}$
c) $36^{t-2} = 216^{-2t}$
d) $8^{2x-1} = 16^{x-1}$
e) $25^{1-3x} = 125^{-x}$
f) $16^{3+k} = 32^{1-2k}$

9. Solve and check.

a) $5 = 25^{\frac{x}{2}}$
b) $8 = 2^{\frac{x}{3}}$
c) $9^{\frac{x}{5}} = 27$
d) $\frac{1}{2} = 2^{\frac{a}{3}}$
e) $4^{\frac{x}{4}} = \frac{1}{8}$
f) $\left(\frac{3}{2}\right)^{\frac{m}{2}} = \frac{4}{9}$

10. Solve.

a) $3(5^{x+1}) = 15$
b) $2(3^{y-2}) = 18$
c) $5(4^x) = 10$
d) $2(4^{v+1}) = 1$
e) $2 = 6(3^{4y-2})$
f) $27(3^{3x+1}) = 3$

11. Solve and check.

a) $2^{x+2} - 2^x = 48$
b) $4^{x+3} + 4^x = 260$
c) $2^{a+5} + 2^a = 1056$
d) $6^{x+1} + 6^{x+2} = 7$
e) $3^{x+3} - 3^{x+1} = 648$
f) $10^{c+4} + 10^{c+3} = 11$
g) $2^{x+2} - 2^{x+5} = -7$
h) $3^{m+1} + 3^{m+2} - 972 = 0$
i) $5^{n+2} - 5^{n+3} = -2500$

10 a)

$$3(5^{x+1}) = 15$$

$\div 3 \rightarrow$

$$5^{x+1} = 5$$

$$5^{x+1} = 5^1$$

$$\therefore x+1 = 1$$

$$x = 0$$